JOURNAL OF COMBINATORIAL THEORY (B) 21, 245–256 (1976)

Infinite Digraphs Isomorphic with Their Line Digraphs

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Communicated by W. T. Tutte

Received March 19, 1974

In this paper we give a structural characterization of the digraphs that are isomorphic with their line digraphs.

1. INTRODUCTION

The *line digraph* of a digraph D with nonempty edge set is denoted by $L(D)$, has $E(D)$ as its vertex set, and has $(\alpha, \beta) \in E(L(D))$ for $\alpha, \beta \in E(D)$ if and only if $\alpha = (a, b)$ and $\beta = (b, c)$ for some $a, b, c \in V(D)$.

As pointed out in [2] most of the results concerning line digraphs have to do with the following three problems: (a) Characterize the digraphs that are line digraphs, (b) give a class of digraphs such that the line graph transformation is a one-to-one function from this class onto the class of all line digraphs, and (c) characterize those digraphs that are isomorphic to their line digraphs.

Satisfactory solutions to the first two of these problems have been known for some time in the finite case and the extension to the infinite case was straightforward [see 2]. In [3], Hemminger solved the third problem for finite digraphs. This solution was also good for certain infinite digraphs that contain a finite dicycle.

The purpose of this paper is to complete the characterization of digraphs that are isomorphic to their line digraphs.

Our notation is standard and will follow that used in [2, 3].

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2. The Main Results

For the sake of completeness and for reference in Section 3 we will first review the known results [see 3]. This requires a little background material.

There are a number of characterizations of digraphs which are line digraphs, but the most useful is due to Heuchenne [4]: D is a line digraph if and only if whenever the (directed) edges (a, c), (b, c), and (b, d) appear, so does the edge (a, d). Hemminger [2] called this the first Heuchenne condition and defined the n'th Heuchenne condition to be that if there are openly disjoint dipaths of length n from a to c, from b to c, and from b to d, then there must also be one from a to d that is openly disjoint from the others. It follows that if $L(D) \simeq D$ then D must satisfy all Heuchenne conditions. Before stating the theorem we state three results from [2, 3].

(1) If $L(D) \simeq D$, then each weakly connected component of D has at most one finite dicycle.

(2) If D has just one finite dicycle Z, each vertex v_i of Z has an associated subgraph $A_{v_i}(Z)$ (or A_i for short) of $D - E(Z)$, induced by those vertices which can reach v_i as well as a subgraph $C_{v_i}(Z)$ (or C_i for short) of $D - E(Z)$ induced by those which v_i can reach. Since D has just the one finite dicycle, the only A_i that can intersect C_j is A_j and then they only have v_j in common.

(3) If $L(D) \simeq D$, the subgraphs A_i must be pairwise disjoint arborescences, and similarly the subgraphs C_i must be pairwise disjoint counterarborescences. (This observation follows immediately from Lemma 1.)

Note. We are continuing to use the terms arborescence and counter-arborescence as in the papers dealing with this subject—however, this is a switch of the definitions as originally given by Berge [1].

THEOREM 1. *Let D_0 be a digraph which consists of a finite dicycle with vertices $v_1, v_2, \ldots, v_r$ together with pairwise disjoint arborescences A_i and counterarborescences C_i at its vertices. Let D be the digraph generated by D_0 and the Heuchenne conditions (see the following definition of "Heuchenne completion" for further clarification) and let a and b be the minimum positive integers for which, respectively, $A_i \simeq A_{i+a}$ and $C_i \simeq C_{i+b}$ for all i. Then D has period t if and only if g.c.d. $(a, b) = t$. (D has period t if t is the smallest positive integer k such that $L^{m+k}(D) \simeq L^m(D)$ for some $m \geqslant 0$. Note that D is likely disconnected.)*

It is our goal to obtain a similar result for period one digraphs in which there is no finite dicycle (such a digraph must of course be infinite).

We will divide these into two subclasses: The first class consists of those that contain a two-way infinite dipath, called an *infinite dicycle* (using terminology similar to "infinite cyclic groups"); and the second class consists of those that contain no dicycles, finite or infinite. To simplify the statement of the theorems we make the following definitions.

DEFINITION. A digraph B is called *basic* if it is a tree (i.e., has no semicycles) having not both a source and a sink, and which consists of an infinite dicycle $P = (..., v_{-1}, v_0, v_1, ...)$ together with arborescences A_m and counterarborescences C_m rooted at v_m such that either

(1) every A_m is trivial or every C_m is trivial, or

(2) there exist nontrivial A_i and C_j and there is an integer n such that for all m, $C_{m+n} \simeq C_m$ and $A_{m+n+1} \simeq A_m$.

Let B be a basic digraph as in the definition. We will now describe an extension of B to a digraph D that will satisfy the Heuchenne conditions.

Since B is a tree the A_i's are disjoint, the C_i's are disjoint, and $A_i \cap C_j = \{v_i\} \cap \{v_j\}$.

For a given m, let $E(A_m, k) = \{(u, v) \in A_m : d(u, v_m) = k\}$, $V(A_m, k) = \{u \in A_m : d(u, v_m) = k\}$, $E(C_m, k) = \{(u, v) \in C_m : d(v, v_m) = k\}$, and let $V(C_m, k) = \{u \in C_m : d(u, v_m) = k\}$; i.e., $E(A_m, k)$ is the set of edges of A_m at distance k from v_m and $V(A_m, k)$ is the set of vertices of A_m at distance k from v_m. Similarly for $E(C_m, k)$ and $V(C_m, k)$.

Now, for each m, extend B by adding all edges of the form (u, v) with $u \in V(A_{m+n+1}, 1)$ and $v \in V(C_{m+n}, 1)$ and let $B_{m,1}$ be the subdigraph induced by these edges plus the edges from $E(C_{m+n}, 2) \cup E(A_{m+n+1}, 2)$. Further extend B by adding the line digraph of $B_{m,1}$ to B by using $L(E(C_{m+n}, 2)) = V(C_{m+2n}, 2)$ and $L(E(A_{m+n+1}, 2)) = V(A_{m+2n+2}, 2)$. Let $B_{m,2}$ be the subdigraph induced by the edges of $L(B_{m,1})$ plus the edges of $E(C_{m+2n}, 3) \cup E(A_{m+2n+2}, 3)$. Continue extending B in this manner by induction and let D be the resulting digraph.

Note that D is weakly connected (because of the no source or no sink restriction) and that the Heuchenne conditions hold.

DEFINITION. For a basic digraph B we will denote the graph obtained above by $H(B)$ and call it the *Heuchenne completion of B*.

The reason for continually extending the modified basic digraph by the line digraph of the part that last modified it is that in the proof of the necessity of the condition in Theorem 2 we will have an isomorphism $\tau: L(D) \simeq D$ that takes the line graph image of a basic digraph B of D

onto B. Thus $L(B)$, $L^2(B)$,... will be isomorphic to subdigraphs of D that contain B as in the Heuchenne completion of B.

The second and third digraphs in Fig. 1 illustrate the first two steps in forming the Heuchenne completion of the basic digraph given there.

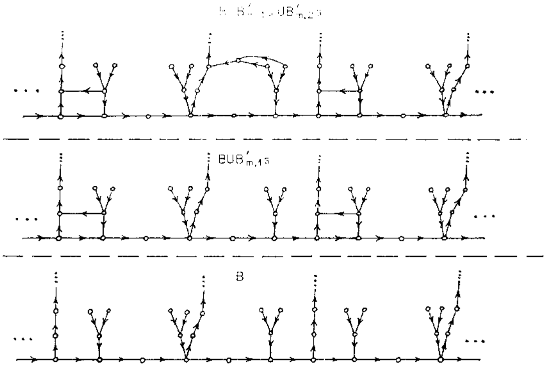

FIGURE 1

THEOREM 2. *Let D be a weakly connected digraph with infinite dicycles, but with no finite dicycles. Then $L(D) \simeq D$ if and only if there is a basic digraph B such that $D \simeq H(B)$.*

Proof. If B is a basic digraph, then $H(B) = B$ if and only if either all the A_m's are trivial or all the C_m's are trivial.

The condition is clearly sufficient in that case so let B be a basic digraph as in the definition and let $\rho_m : C_m \simeq C_{m+n}$ and $\sigma_m : A_m \simeq A_{m+n+1}$ be the given isomorphisms. Define $\tau \colon E(B) \to V(B)$ as follows: $\tau(e_m) = v_{m+n+1}$ for all m where $e_m = (v_m , v_{m+1})$, $\tau(u, v) = \sigma_m(u)$ for all m and all $(u, v) \in E(A_m)$, and $\tau(u, v) = \rho_m(v)$ for all m and all $(u, v) \in E(C_m)$. We will now show that τ has a natural extension to an isomorphism τ' of $L(H(B))$ onto $H(B)$.

Let $B_{m,k}$, $k \geqslant 1$, be as in the definition of $H(B)$. For $e \in E(B_{m,k})$ define $\tau'(e)$ to be the vertex e of $L(B_{m,k})$; i.e., $\tau'(e)$ is a vertex of $B_{m,k+1}$. By defining $\tau'(e) = \tau(e)$ for $e \in E(B)$ we have obviously defined τ' as a one-to-one function of $E(H(B))$ onto $V(H(B))$ that extends τ. Moreover, two edges e and f of $H(B)$ are head-to-tail if and only if $(\tau'(e), \tau'(f)) \in E(H(B))$; this clearly holds for τ on B and so it holds for τ' on $H(B)$ since τ' was extended in a line digraph manner. It follows that $\tau' \colon L(H(B)) \simeq H(B)$.

Because the proof of the necessity of the condition is rather long we will break it into a series of lemmas. Throughout this portion of the proof we will let $\tau \colon L(D) \simeq D$ denote the given isomorphism. Thus τ is a one-to-one mapping of $E(D)$ onto $V(D)$ such that edge e_1 is adjacent *to* edge e_2 if and only if $\tau(e_1)$ is adjacent *to* $\tau(e_2)$.

LEMMA 1. *There is at most one dipath between any pair of vertices of D.*

Proof. Suppose there are two openly disjoint dipaths, $(u, a_1, a_2, ..., a_p, v)$ and $(u, b_1, b_2, ..., b_q, v)$, between u and v. If $1 \leqslant p \leqslant q$, then between the head of edge $\tau^{-1}(u)$ and the tail of edge $\tau^{-1}(v)$ there are two openly disjoint dipaths of lengths $p - 1$ and $q - 1$ respectively. Continuing in this way we are led to vertices u' and v', a dipath $(u', c_1, c_2, ..., c_{q-p+1}, v')$, and an edge (u', v') different from the dipath (there is a third point c_1 in the dipath since a line digraph does not have multiple edges). But then, by the Heuchenne condition, there is an edge $(c_{q-p+1}, c_1) \in E(D)$. Thus $(c_1, c_2, ..., c_{q-p+1}, c_1)$ is a dicycle (perhaps a loop) in D and we have a contradiction.

LEMMA 2. *There is a dipath containing both an edge e and its image vertex* $\tau(e)$.

Proof. Suppose there is no such dipath. Since D is weakly connected there is such a semipath; let P be a shortest one. Then P consists of dipaths $P_1, P_2, ..., P_k$, $k > 1$, with alternating orientations where the first edge of P_1 is e and the last vertex of P_k (and hence of P) is $\tau(e)$. By converse duality (the converse is obtained by reversing the orientation on all edges) we can assume that the first vertex of P_1 is u where $e = (u, v)$.

Suppose that P_k is oriented away from $\tau(e)$, say $f = (\tau(e), x)$ is the last edge of P_k. Then, since $L(D) \simeq D$, there is an edge $x' = (v, w)$ such that $\tau(x') = x$. But this is impossible because the semipath with edge set $\{x'\} \cup E(P_1) - \{e\} \cup E(P_2) \cup \cdots \cup E(P_{k-1}) \cup E(P_k) - \{f\}$ is shorter than P and contains the edge x' and its image vertex $\tau(x') = x$.

Suppose that P_k is oriented towards $\tau(e)$. Then, since $L(D) \simeq D$, there is a dipath Q_k, oriented towards its end vertex u, whose edges map, under τ, to the vertices of P_k. Thus the semipath consisting of Q_k, $P_1, P_2, ..., P_{k-1}$ is of the same length as P, has Q_k oriented away from its first vertex, has P_{k-1} oriented away from its last vertex, and has its first edge mapping, under τ, to its last vertex. In the last paragraph we saw the impossibility of this state of affairs. That completes the proof of Lemma 2.

By Lemma 2, there is an edge e and a dipath P_e that has e as an endedge and $\tau(e)$ as an endvertex. By converse duality, we can assume that $\tau(e)$ is ahead of e on P_e. Thus we let P_e have vertex representation $(v_0, v_1, ..., v_{n+1})$ with $e = e_0 = (v_0, v_1)$ and $\tau(e_0) = v_{n+1}$, $n \geqslant 0$. Then there is a dipath with consecutive edges $e_{-n-1}, e_{-n}, ..., e_0$ such that $\tau(e_m) = v_{m+n+1}$, $-n - 1 \leqslant m \leqslant 0$. Since D has no finite dicycles, repeating this procedure by induction results in dipaths whose union is a one-way infinite dipath $Q = (..., v_{-2}, v_{-1}, v_0, ..., v_{n+1})$ with $\tau(e_m) = v_{m+n+1}$, $m \leqslant 0$, where $e_m = (v_m, v_{m+1})$. Let M be the maximal counter-arborescence of D that has root v_1.

LEMMA 3. *If $n = 0$, i.e., if $\tau(e_m) = v_{m+1}$ for all $m \leqslant 0$, then there is an isomorphism σ: $L(D) \simeq D$ such that σ takes each edge of $Q \cup M$ to its head.*

Proof. As before we will say that an edge of M is at distance k from v_1 if the directed distance of the head of the edge from v_1 is k and we let $E(M, k)$ denote the set of such edges. Similarly we let $V(M, k)$ denote the set of vertices of M at distance k from v_1 in M. Since $\tau(e_0) = v_1$ we have, by induction, that

(1) $\tau(E(M, k)) = V(M, k)$ for all $k \geqslant 1$.

It follows that $A_{v_1}(Q \cup M)$ (recall that this is the union of the dipaths into v_1 that include no other vertices of $Q \cup M$) generates all of the $A_v(Q \cup M)$, $v \in V(M)$, under iteration of τ. If $A_{v_1}(Q \cup M)$ is trivial, then all the $A_v(Q \cup M)$ are trivial and the lemma follows easily. In any case we have

(2) $A_u(Q \cup M) \simeq A_v(Q \cup M)$ for all $u, v \in V(M)$.

Hence, if we let $h(e)$ denote the head of the edge e, we have, for each $e \in E(M)$, an isomorphism $\rho_e : A_{\tau(e)}(Q \cup M) \simeq A_{h(e)}(Q \cup M)$. Furthermore, by (1) and Lemma 1, we have

(3) $A_u(Q \cup M) \cap A_v(Q \cup M) = \varnothing$ if $d(v_1, u) < d(v_1, v)$.

For otherwise, by repeated use of τ^{-1}, this would lead to a similar situation in which there was a dipath in M from u to v which means that there are two distinct dipaths between u and v contrary to Lemma 1.

Since the $A_v(Q \cup M)$ for $v \in V(M)$ are generated by $A_{v_1}(Q \cup M)$ under iteration of τ, we see that

(4) $A_u(Q \cup M)$ and $A_v(Q \cup M)$ are k-hinged if $d(v_1, u) = d(v_1, v)$ and $d(v_1, u) - k$ is the distance from v_1 to the vertex where the dipaths from v_1 to u and v separate.

(5) If e is an edge oriented out of $A_v(Q \cup M)$ for $v \in V(M, m)$ (i.e., the tail of e is in $A_v(Q \cup M)$ but the head is not), then e is either an edge of M or the tail of e is in $V(A_v(Q \cup M), k)$ for some $k > m$.

For suppose that $e = (x, y) \notin E(M)$ and that $d(x, v) \leqslant m$. Then, by repeated application of τ^{-1} we would have this situation with $1 = d(x, v) \leqslant m$. But then, using τ^{-1} one more time, we would have

$\tau^{-1}(x) = (x_1, x_2) \in E(D)$ with $x_2 \in V(M, d(v_1, v) - 1)$. But this means that $\tau^{-1}(y) \in E(M)$, and hence that $y \in V(M)$—which is a contradiction.

Note the consequence, by (4), that if $e \notin E(M)$, as in (5), then $t(e)$ (the tail of e) is in all of the $A_u(Q \cup M)$, $u \in V(M, m)$.

We now define a function $\sigma: E(D) \to V(D)$ as follows:

(a) $\sigma(e) = h(e)$ if $e \in E(M)$,

(b) $\sigma(e) = \rho_f \tau(e)$ if $e \in V(A_{t(f)}(Q \cup M), k)$, $k \leqslant d(v_1, t(f))$, $f \in E(M)$,

(c) $\sigma(e) = \tau(e)$ otherwise.

Using the properties enumerated above one easily checks that σ is a well-defined, one-to-one function of $E(D)$ onto $V(D)$ that preserves adjacency, i.e., σ is the desired isomorphism of $L(D)$ onto D.

LEMMA 4. *There is a nonnegative integer n, an infinite dicycle $P = (\ldots, v_{-1}, v_0, v_1, \ldots)$, and an isomorphism $\sigma: L(D) \simeq D$ such that $\sigma(e_m) = v_{m+n+1}$ for all m where $e_m = (v_m, v_{m+1})$.*

Proof. We use the n obtained immediately preceding Lemma 3. Suppose that $n > 0$. Let $v_{n+2} = \tau(e_1)$. Since $n > 0$ and since D has no finite dicycles, $e_{n+1} = (v_{n+1}, v_{n+2})$ is an edge of D not in Q. Extend Q to a new dipath by adding v_{n+2} and e_{n+1}. Repeating this procedure by induction results in dipaths whose union is an infinite dicycle $P = (\ldots, v_{-1}, v_0, v_1, \ldots)$ with edges $e_m = (v_m, v_{m+1})$ such that $\tau(e_m) = v_{m+n+1}$. The lemma follows with $\sigma = \tau$ in this case.

Suppose that $n = 0$. Then the lemma follows from Lemma 3 if M has an infinite one-way dipath out of v_1. In fact, it follows, by the same technique, if there is a one-way infinite dipath out of any vertex of Q. But that must be the case; for suppose otherwise. Then no infinite dicycle of D intersects Q, hence, by the weak connectivity, there is one closest to Q; call it R. But then $\sigma^{-1}(R)$ is an infinite dicycle closer to $\sigma^{-1}(Q)$ than R is to Q which is a contradiction since $\sigma^{-1}(Q)$ is Q.

That completes the proof of the lemma and we are now ready to complete the proof of the theorem. To do so, let σ and P be as in Lemma 4, let $A_m = A_{v_m}(P)$, $C_m = C_{v_m}(P)$, and let B be the union of P, the A_m's, and the C_m's. That the A_m's are arborescences and that the C_m's are counterarborescences follows from Lemma 1 as does the fact that B is a tree. Since $\sigma(e_m) = v_{m+n+1}$ we have $C_{m+n} \simeq C_m$ and $A_{m+n+1} \simeq A_m$ for all m. That B is a basic digraph follows from the connectivity of D; for a weakly connected digraph D with $L(D) \simeq D$ does not have both sources and sinks [4; or see 2, Lemma 3.6].

If B is as in (1) of the definition of a basic digraph, then, by the observation at the beginning of the proof, the theorem follows easily. So we can assume that B is as in (2) in that definition. It follows, by application of σ, that there is an edge from each vertex of $V(A_{m+n+1}, 1)$ to each vertex of $V(C_{m+n}, 1)$. Let $B_{m,1}$ denote the subdigraph induced by these edges. Define $B_{m,2}$ as the subdigraph induced by the vertex set $V(A_{m+2n+2}, 2) \cup V(C_{m+2n}, 2) \cup \sigma(E(B_{m,1}))$ and inductively define $B_{m,k}$ as the subdigraph induced by the vertex set $V(A_{m+kn+k}, k) \cup V(C_{m+kn}, k) \cup \sigma(E(B_{m,k-1}))$. Note that the $B_{m,k}$ are precisely the Heuchenne condition parts of D generated by B.

Suppose that D has some vertex other than those already accounted for by B and the $B_{m,k}$. Then, by the weak connectivity of D, there is an edge e of one of the following forms:

(a) directed out of an arborescence A_m,

(b) directed into a counterarborescence C_m,

(c) directed into some $B_{m,k}$, $k > 1$,

(d) directed out of some $B_{m,k}$, $k > 1$.

We will only examine cases (a) and (c) as (b) and (d) are treated in the same manner.

Case (a). We can assume that $u \in V(A_m, k)$ where $e = (u, v)$ and that k is the smallest such integer associated with an edge of this type. Then the edge $\sigma^{-1}(v)$ is an edge directed out of $V(A_{m-n-1}, k-1)$. But this contradicts the choice of k unless $k = 1$; but then $\sigma^{-1}(v)$ is in C_{m-n-1} and so e is in $B_{m-n-1,1}$.

Case (c). If $e = (u, v)$ with $v \in B_{m,k}$, $k > 1$, and e is not as in case (b), then $\sigma^{-1}(u)$ is such an edge associated with $B_{m,k-1}$ except that $\sigma^{-1}(u)$ might be in an arborescence. By picking an edge for which k is as small as possible we are in fact forced to the conclusion that $\sigma^{-1}(u)$ is in an arborescence. But this means that e is in an arborescence.

We conclude that $D \simeq H(B)$. That completes the proof of Theorem 2.

DEFINITION. Call a digraph B a *restricted basic digraph* if B is a basic digraph with $n = 0$ such that

(1) $C_m(P) = \varnothing$ for all $m > 0$,

(2) $C_m(P)$ contains no infinite dipath for each m, and

(3) if some $A_m(P)$ is nontrivial for $m > 1$, then there is a $k \leqslant 1$ such that $V(C_k, m-k) \neq \varnothing$.

Note that in forming $H(B)$ from a restricted basic digraph B, no edges are added incident with vertices from the sets $V(A_m, k)$ for $m > 1$ and $0 \leqslant k < m$. Denote this last set by $V(A_m, <m)$.

DEFINITION. For a restricted basic digraph B as above let $TH(B)$ be the digraph $H(B) - \bigcup_{m>1} V(A_m, <m)$. We call *$TH(B)$ the truncated Heuchenne completion* of B.

One easily sees that the isomorphism τ' of $L(H(B))$ onto $H(B)$ given in the sufficiency proof of Theorem 2, when restricted to $L(TH(B))$, is an isomorphism of $L(TH(B))$ onto $TH(B)$. And the combination of the no source or no sink condition in the definition of a basic digraph and condition (3) in the definition of a restricted basic digraph insures that $TH(B)$ is weakly connected. Because of condition (2) in the definition of a restricted basic digraph, $TH(B)$ has no infinite dicycle. Thus we have proved the sufficiency of the condition in the following theorem which characterizes our second class of period one digraphs.

THEOREM 3. *Let D be a weakly connected digraph with no dicycles. Then $L(D) \simeq D$ if and only if there is a restricted basic digraph B such that $D \simeq TH(B)$ or $D^r \simeq TH(B)$.*

Proof. Suppose that D is a weakly connected digraph with no dicycles and with $L(D) \simeq D$. The proof continues as in Theorem 2 through the proof of Lemma 3. The situation where $\tau(e)$ is behind e on P_e (following Lemma 2) leads to the part of the theorem involving D^r. The $n > 0$ case cannot occur here. Thus, as a consequence of Lemma 3 and the fact that D has no dicycles, we can assume that $Q = (..., v_{-2}, v_{-1}, v_0, v_1)$ is a dipath with v_1 a sink of D and with σ: $L(D) \simeq D$ such that $\sigma(e_m) = v_{m+1}$, $m \leqslant 0$ where $e_m = (v_m, v_{m+1})$.

Let the edges, other than e_0, that are incident with v_1 be indexed by a set I where I contains no integer and let $A_{1,i}$ be the maximal arborescence containing the edge e_i, $i \in I$, but no e_j, $j \in I \cup \{0\} - \{i\}$. Thus $A_1(Q) = \bigcup_{i \in I} A_{1,i}$ and $\sigma(e_i)$ is a sink in D for all $i \in I$. Moreover $\sigma(A_{1,i})$ is a maximal arborescence in D; denote this arborescence by $A_{2,i}$. Define $A_{m,i}$, $m > 1$, $i \in I$, inductively as the image, under σ, of $A_{m-1,i}$ and denote its root by $a_{m,i}$.

We now extend D to a digraph with an infinite dicycle (see Fig. 2 for an example). First add a one-way infinite dipath $(v_1, v_2, v_3, ...)$ to v_1 where the v_m, $m > 1$ are new vertices and let P be the infinite dicycle $(..., v_{-1}, v_0, v_1, ...)$. For each $i \in I$ and $m > 1$ add a dipath $P_{m,i}$ from $a_{m,i}$ to v_m of length $m - 1$. Call this new digraph D'.

Extend σ to a function σ' on D' by mapping each edge of P to its head and by mapping the edges of $P_{m,i}$ to the vertices of $P_{m+1,i}$ in the natural dipath order.

It is obvious from the definitions that σ' is an isomorphism of $L(D')$ onto D' since σ was an isomorphism of $L(D)$ onto D. The weak connectivity of D' follows immediately from that of D.

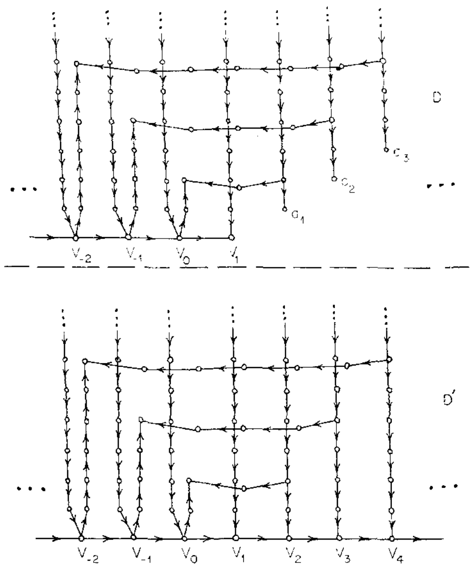

FIGURE 2

Summarizing: D' is a weakly connected digraph with an infinite dicycle P and with an isomorphism σ' of $L(D')$ onto D' that takes each edge of P onto its head. If we let B be the basic digraph consisting of P and the arborescences and counterarborescences rooted at P, then it is clear that B is a restricted basic digraph (condition (3) in that definition follows from the weak connectivity of D'). Continuing as in the proof of Theorem 2, we conclude that $D' \simeq H(B)$. But, since the new edges added to B in forming $H(B)$ are incident only to vertices of D' that are in D, we see that $D \simeq TH(B)$, which is what we wanted to prove.

3. THE DISCONNECTED CASE

This case is easily related to the weakly connected case. To do so let two edges of D be A-related if there is a path (not necessarily a dipath) with these two edges as their endedges and with no source or sink of D as an interior vertex of the path. It follows that A is an equivalence relation on $E(D)$. We call the subdigraphs induced by the equivalence classes the A-components of D. It follows that two A-components of D have at most sources and sinks in common and that if π: $L(D) \simeq D$, then π maps A-components of D onto weakly connected components of D. An A-component is called an atom if it contains an edge e such that $\pi(e)$

is also in that A-component. Thus, by the last observation, the image of an atom under π is the weakly connected component containing it. Thus a weakly connected component contains at most one atom and we see from the proof of Lemma 2 that it is precisely the defining property of an atom that we need to validate Lemma 2 for weakly connected components that contain an atom. If B is the atom of a weakly connected component C, then it is clear (though a little involved to check the details) that if C does not contain a finite dicycle, then C is either the Heuchenne completion or the truncated Heuchenne completion of B and that B is an *almost basic* or an *almost restricted basic* digraph where these concepts are obtained from those of basic and restricted basic digraphs by dropping the no source or no sink condition (whose only role was to insure weak connectivity).

Let $C = C_0$. Then $L(C_0)$ is the union of weakly connected components of D (including C_0), say C_0 and $C_{1,i}$, $i \in I_1$. Clearly none of the $C_{1,i}$'s contain atoms so $L(\bigcup_{i\in I_1} C_{1,i})$ is also the union of weakly connected components of D, say $C_{2,i}$, $i \in I_2$, and these are disjoint from C_0 and the $C_{1,i}$'s. Continuing by induction we obtain a set $\mathscr{C} = \{C_0\} \cup \{C_{j,i} : j \geqslant 1, i \in I_j\}$ of weakly connected components of D such that if D_0 is the subdigraph of D induced by the components in $\mathscr{C}$, then $L(D_0) \simeq D_0$. We call D_0 the subdigraph of D *generated by* C_0. See Fig. 3 for an illustration of this.

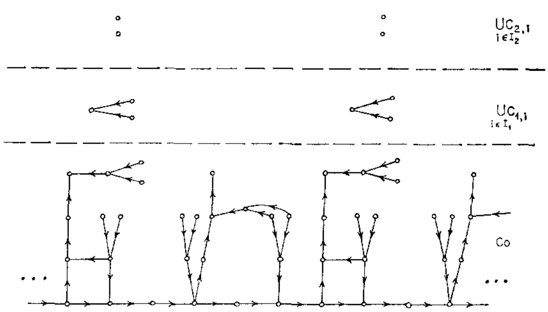

FIGURE 3

Now consider a weakly connected component C that is in no subdigraph of D generated by a weakly connected component containing an atom. Then C is a subdigraph of $L(C_{-1})$ for some atomless weakly connected component C_{-1}. Continuing to run these backwards by induction and then run each obtained component forward via L we get a set $\mathscr{C}' = \{C_{j,i} : j \in Z, i \in I_j\}$ of atomless weakly connected components of D such that

$L(\bigcup_{i\in I_j} C_{j,i}) = \bigcup_{i\in I_{j+1}} C_{j+1,i}$ and such that $L(D') \simeq D'$ where D' is the subgraph of D induced by the components in $\mathscr{C}'$.

That completes the description of digraphs that are isomorphic to their line digraphs.

REFERENCES

1. C. BERGE, "The Theory of Graphs and Its Applications," Methuen, London, 1962.
2. R. L. HEMMINGER, Line digraphs, *in* "Graph Theory and Applications," (Y. Alavi, D. R. Lick, and A. T. White, Eds.), pp. 149–164, Springer-Verlag, Berlin, 1972.
3. R. L. HEMMINGER, Digraphs with periodic line digraphs, *Studia Sci. Math. Hungar.* **9** (1974), 27–31.
4. C. HEUCHENNE, Sur une certaine correspondence entre graphes, *Bull. Soc. Roy. Sci. Liege* **33** (1964), 743–753.